INTEGRATION BY AUXILIARY INTEGRALS.

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Let it now be required to integrate

$$dy = \frac{dx}{\sqrt{(\text{tg}\, x)}}.$$

We know from our elementary forms that when

$$du = \frac{2x.dx}{1+x^4}, \text{ then}$$

$$u = \text{arc tg}\, x^2;\ \therefore x = \sqrt{(\text{tg}\, u)}.$$

On the other hand, when

$$dv = \frac{dx}{1+x^4}$$

v is a well known integral. Now

$$\frac{dv}{du} = \frac{1}{2x} = \frac{1}{2\sqrt{(\text{tg}\, u)}};\ \therefore dv = \frac{du}{2\sqrt{(\text{tg}\, u)}}.$$

But $v = \frac{1}{4\sqrt{2}} \lg\left(\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}\right) + \frac{1}{2\sqrt{2}} \text{arc tg}\left(\frac{x\sqrt{2}}{1-x^2}\right)$, hence

$$\int \frac{du}{\sqrt{(\text{tg}\, u)}} = \frac{1}{2\sqrt{2}} \lg\left(\frac{1+\sqrt{(2.\text{tg}\, u)}+\text{tg}\, u}{1-\sqrt{(2.\text{tg}\, u)}+\text{tg}\, u}\right) + \frac{1}{\sqrt{2}} \text{arc tg}\left(\frac{\sqrt{(2.\text{tg}\, u)}}{1-\text{tg}\, u}\right).$$

This last equation, in connection with our elementary integrals, leads to a number of new integrals. Thus, for example take

$$du = \frac{dx}{\sqrt{(2a-x^2)}},\ \therefore u = \tfrac{1}{2}\text{arc sin}\ \frac{x^2-a}{a},\ \therefore \text{tg}\, 2u = \frac{x^2-a}{x\sqrt{(2a-x^2)}},$$

$$\therefore \frac{du}{\sqrt{(\text{tg}\, 2u)}} = \frac{dx}{\sqrt{(2a-x^2)}} \cdot \frac{x^{\frac{1}{2}}(2a-x^2)^{\frac{1}{4}}}{\sqrt{(x^2-a)}};\ \text{hence}$$

$$\int \frac{\sqrt{x}\ dx}{(2a-x^2)^{\frac{1}{4}}(x^2-a)^{\frac{1}{2}}} = \int \frac{du}{\sqrt{(\text{tg}\, 2u)}}.$$

Taking the same integral in connection with the elementary form:

$$dy = \frac{2xdx}{\sqrt{(1-x^4)}},\ \therefore y = \text{arc sin}\, x^2,\ \therefore x^2 = \sin y,\ \therefore \text{tg}\, y = \frac{x^2}{\sqrt{(1-x^4)}}$$

$$\frac{dy}{\sqrt{(\text{tg}\, y)}} = \frac{2xdx}{\sqrt{(1-x^4)}} \cdot \frac{(1-x^4)^{\frac{1}{4}}}{x} = \frac{2dx}{(1-x^4)^{\frac{1}{4}}};\ \text{hence}$$

$$\int \frac{dx}{(1-x^4)^{\frac{1}{4}}} = \tfrac{1}{2}\int \frac{dy}{\sqrt{(\text{tg}\, y)}}.$$

Taking the same integral in connection with the elementary form:

$$dy = \frac{dx}{(2x^2+1)\sqrt{(x^2+1)}}, \therefore y = -\tfrac{1}{2} \text{ arc tg} \sqrt{1+\frac{1}{x^2}},$$

$$\therefore \sqrt{\frac{x^2+1}{x^2}} = -\text{tg } 2y, \therefore i\sqrt{(\text{tg } 2y)} = \frac{(x^2+1)^{\frac{1}{4}}}{\sqrt{x}};$$

$$\frac{dy}{i\sqrt{(\text{tg } 2y)}} = \frac{dx}{(2x^2+1)\sqrt{(x^2+1)}} \cdot \frac{\sqrt{x}}{(x^2+1)^{\frac{1}{4}}}, \text{ hence}$$

$$\int \frac{\sqrt{x}.dx}{(2x^2+1)(x^2+1)^{\frac{3}{4}}} = -i \int \frac{dy}{\sqrt{(\text{tg } 2y)}}.$$

Before proceeding to the further employment of our elementary integrals for the integration of other functions, I propose to demonstrate the fruitfulness of our method by showing that all (or nearly all) the integrals known in finite form and contained in the larger collections of integrals are found by our method without resorting to the decomposition into partial fractions and without the process of rationalization. For these integrations the twelve or fifteen fundamental integrals of the text-books, when used as auxiliary integrals, suffice. It will be seen that our results are different in form from those found by the ordinary method, and at the same time better adapted to numerical computations, inasmuch as they allow of more extensive use of the tables of logarithms and of circular functions. I shall follow the arrangement of *Minding's* "Integraltafeln".

I°. Rational Functions.

The first six tables of *Minding* contain the functions of the form

$$dv = \frac{x^m dx}{(a+bx)^n},$$

where m and n are positive integers, m from 1 to 6 and n from 1 to 10. In all these cases we take

$$du = \frac{dx}{a+bx}; \therefore u = \frac{1}{b} \lg (a+bx).$$

$$\therefore a + bx = e^{bu}; \quad x = (1 \div b)(e^{bu} - a).$$

This gives

$$\frac{dv}{du} = \frac{x^m}{(a+bx)^{n-1}} = \frac{1}{b^m} \frac{(e^{bu}-a)^m}{e^{(n-1)bu}},$$

the integration of which presents no difficulty. In order to apply this to a special example, let it be required to integrate

$$dv = \frac{x^5 dx}{(a+bx)^8}, \text{ then } \frac{dv}{du} = \frac{x^5}{(a+bx)^7} = \frac{1}{b^5} \frac{(e^{bu}-a)^5}{e^{7bu}},$$

of which the integration is so simple as to require no further remarks; and

the lists of integrals of the above form already calculated become needless. This result, it is readily seen, is likewise found by the substitution $a+bx=u$.

Tables VII to XIII (of *Minding*) contain the functions of the form

$$dv = \frac{dx}{x^m(a+bx)^n},$$

where m and n are positive whole numbers, m from 1 to 7 and n from 1 to 8. In all these cases we take

$$du = \frac{dx}{x(a+bx)}; \therefore u = -\frac{1}{a}\lg\left(\frac{a+bx}{x}\right);$$

$$\therefore a+bx = \frac{a}{1-b.e^{au}}; \quad x = \frac{a.e^{au}}{1-b.e^{au}}.$$

This gives

$$\frac{dv}{du} = \frac{1}{x^{m-1}(a+bx)^{n-1}} = \frac{(1-b.e^{au})^{m-1}}{a^{m-1}.e^{(m-1)au}}.\frac{(1-b.e^{au})^{n-1}}{a^{n-1}},$$

which is integrable without any difficulty. By way of an example of this kind, let it be required to integrate

$$dv = \frac{dx}{x^6(a+bx)^8}, \text{ then } \frac{dv}{du} = \frac{1}{x^5(a+bx)^7};$$

$$\therefore dv = \frac{1}{a^{12}}\frac{(1-b.e^{au})^{12}}{e^{5au}}du.$$

Here the integration is so simple as to require no further remark. Again it appears that the tables of integrals already calculated become superfluous when our method is employed; whereas the integration by the ordinary method of decomposition in such an example as we have just treated would involve considerable labor.

Tables XIV to XIX contain integrals of the form

$$v = \int\frac{x^{2m}dx}{(a+bx^2)^n},$$

where m and n are positive whole numbers, m from 1 to 5 and n from 1 to 8. In all these cases we take

$$du = \frac{dx}{a+bx^2}; \therefore u = \frac{1}{\sqrt{(ab)}}\text{ arc tg}\left(x\sqrt{\frac{b}{a}}\right);$$

$$\therefore a+bx^2 = a.\sec^2[\sqrt{(ab)}.u]; \quad x = \sqrt{(a\div b)}.\text{tg}[\sqrt{(ab)}.u].$$

This gives

$$\frac{dv}{du} = \frac{x^{2m}}{(a+bx^2)^{n-1}} = \left(\frac{a}{b}\right)^m.\text{tg}^{2m}[\sqrt{(ab)}.u].\frac{1}{a^{n-1}}.\cos^{2(n-1)}[u\sqrt{(ab)}].$$

For example, let it be required to integrate

$$dv = \frac{x^8dx}{(a+bx^2)^6}; \therefore \frac{dv}{du} = \frac{x^8}{(a+bx^2)^5};$$

$$\therefore\ dv = \frac{1}{b^4 a}\sin^8[u\sqrt{(ab)}]\cos^2[u\sqrt{(ab)}]\,du,$$

the integration of which presents no difficulty.

Now let b be negative, so that (putting a^2 and b^2 as constants) it is req'd to integrate:

$$dv = \frac{x^{2m}dx}{(a^2 - b^2x^2)^n}.$$

In this case we take

$$du = \frac{dx}{a^2 - b^2x^2},\ \therefore\ u = \frac{1}{2ab}\lg\left(\frac{a+bx}{a-bx}\right);$$

$$\therefore\ x = \frac{a}{b}\left(\frac{e^{2abu}-1}{e^{2abu}+1}\right).$$

For our present purpose it is more convenient to employ the equivalent geometric functions; this gives

$$x^2 = -\frac{a^2}{b^2}\operatorname{tg}^2(i.abu);\ a^2 - b^2x^2 = a^2.\sec^2(i.abu).$$

Let it be required, for example, to integrate

$$dv = \frac{x^4dx}{(a^2 - b^2x^2)^6};\ \therefore\ \frac{dv}{du} = \frac{x^4}{(a^2 - b^2x^2)^5};$$

$$dv = \frac{1}{b^4a^6}\sin^4(i.abu)\cos^6(i.abu)\,du,$$

the integration of which presents no difficulty.

Tables XX to XXVI contain the integrals of the form

$$v = \int\frac{dx}{x^{2m}(a+bx^2)^n},$$

where m and n are positive integers, m from 1 to 5 and n from 1 to 7. In all these cases we may again take for auxiliary integral

$$u = \int\frac{dx}{a+bx^2} = \frac{1}{\sqrt{(ab)}}\operatorname{arc\,tg}\left(x\sqrt{\frac{b}{a}}\right);$$

$$\therefore\ x = \sqrt{(a \div b)}\operatorname{tg}[u\sqrt{(ab)}];\ a+bx^2 = a\sec^2[u\sqrt{(ab)}].$$

Let it be required to integrate, for example,

$$dv = \frac{dx}{x^6(a+bx^2)^3};\ \therefore\ \frac{dv}{du} = \frac{1}{x^6(a+bx^2)^2};$$

$$dv = \frac{b^3}{a^5}\cdot\frac{\cos^{10}[u\sqrt{(ab)}]}{\sin^6[u\sqrt{(ab)}]},$$

and the integration presents no difficulty. I omit for the present the forms

$$\int\frac{x^m dx}{(a+bx^n)^p}\ \text{and}\ \int\frac{dx}{x^m(a+bx^n)^p},$$

as we shall have to treat them in connection with auxiliary integrals not now to be employed.

Tables LII to LVI of Minding treat of the forms

$$v = \int \frac{x^m dx}{(a + bx + cx^2)^n},$$

where m and n are positive whole numbers, m from 0 to 6 and n from 1 to 6. In all these cases we take

$$du = \frac{dx}{a+bx+cx^2}; \therefore u = \frac{2}{\sqrt{(4ac-b^2)}} \operatorname{arc\,tg}\left(\frac{\sqrt{(2cx+b)}}{\sqrt{(4ac-b^2)}}\right)$$

so long as $4ac-b^2$ is positive. This gives

$$2cx+b = \sqrt{(4ac-b^2)}\,\mathrm{tg}\,[u\sqrt{(4ac-b^2)}] = a.\mathrm{tg}\,(ua);$$

$$x = \frac{1}{2c}[a.\mathrm{tg}\,(ua) - b];\ a + bx + cx^2 = \frac{a^2}{4c\,.\cos^2(au)}.$$

Let it be required, for example, to integrate

$$v = \int \frac{x^6 dx}{(a+bx+cx^2)^3}, \text{ then}$$

$$\frac{dv}{du} = \frac{x^6}{(a+bx+cx^2)^2} = \frac{1}{(2c)^6}[a.\mathrm{tg}\,(au) - b]^6 \frac{1}{a^4}[4c.\cos^2(au)]^2;$$

$$v = \frac{(4c)^2}{(2c)^6 a^4} \int [a.\mathrm{tg}\,(au) - b]^6 \cos^4(au).du,$$ the integration of which needs no discussion.

When m is negative, so that

$$v = \int \frac{dx}{x^m(a+bx+cx^2)^n}$$

is required, the substitution $x = (1 \div t)$ gives

$$du = -\frac{dt}{at^2+bt+c}; \therefore u = \frac{2}{\sqrt{(4ac-b^2)}} \operatorname{arc\,tg}\left(\frac{2c+bt}{t\sqrt{(4ac-b^2)}}\right);$$

$$t = \frac{2c}{a.\mathrm{tg}\,(\frac{1}{2}a.u) - b}; \therefore \frac{c+bt+at^2}{t^2} = \frac{a^2}{2c}\sec^2(au),$$

which gives very simple results where $m > 2n$ or $m = 2n$.

II. Irrational Functions.

The number of irrational functions integrable in finite form and found in the collections of integrals is quite small. The chief difficulty in the way is the rationalization of the functions, which is often attempted in vain. We will integrate without resorting to rationalization. *Minding* begins with integrals of the form

$$v = \int \frac{x^m(a+bx)^n dx}{\sqrt{(a+bx)}},$$

where m and n are positive integers.

In all these cases we take

$$du = \frac{dx}{\sqrt{(a+bx)}};\ \therefore u = \frac{2}{b}\sqrt{a+bx};$$

$$\therefore a + bx = \tfrac{1}{4}b^2u^2;\ x = \tfrac{1}{4}bu^2 - (a \div b).$$

This gives $dv \div du = x^m(a+bx)^n$. Therefore

$$v = \int \left(\frac{bu^2}{4} - \frac{a}{b}\right)^m \left(\frac{b^2u^2}{4}\right)^n du,$$

the integration of which is exceedingly simple.

When n is negative the same process applies without modification. Thus

when $$v = \int \frac{x^m dx}{(a+bx)^n \sqrt{(a+bx)}},$$ then

$$\frac{dv}{du} = \frac{x^m}{(a+bx)^n} = \left(\frac{bu^2}{4} - \frac{a}{b}\right)^m \div \left(\frac{b^2u^2}{4}\right)^n,$$

which presents no more difficulty than the foregoing form.

When x^m occurs in the denominator, we take

$$du = \frac{dx}{x\sqrt{(a+bx)}};\ \therefore u = \frac{1}{\sqrt{a}} \lg\left(\frac{\sqrt{(a+bx)} - \sqrt{a}}{\sqrt{(a+bx)} + \sqrt{a}}\right);$$

$$\therefore \sqrt{(a+bx)} = \sqrt{a}\left(\frac{1+e^{u\sqrt{a}}}{1-e^{u\sqrt{a}}}\right).$$

It is more convenient, however, to operate with the equivalent goniometric functions, and we have

$$\sqrt{(a+bx)} = i.\sqrt{a}.\mathrm{tg}(i.\tfrac{1}{2}u\sqrt{a});\ x = -(a \div b)\sec^2(i.\tfrac{1}{2}u\sqrt{a}),$$

and the integration of

$$dv = \frac{dx}{x^m(a+bx)^n\sqrt{(a+bx)}}$$

is easily performed in terms of u.

We now come to the form

$$v = \int \frac{x^m dx}{(a+bx+cx^2)^n\sqrt{(a+bx+cx^2)}},$$

where m and n again are positive integers. Here we take the aux. integral

$$u = \int \frac{dx}{\sqrt{(a+bx+cx^2)}} = -\frac{i}{\sqrt{c}} \text{ arc sin}\left(i.\frac{2cx+b}{\sqrt{(4ac-b^2)}}\right).$$

This gives

$$2cx+b = -i\sqrt{(4ac-b^2)}\sin(i.u\sqrt{c}) = -i.\beta\sin(i.u\sqrt{c});$$

$$x = -\frac{1}{2c}[i.\beta\sin(i.u\sqrt{c})+b];\ \sqrt{(a+bx+cx^2)} = \sqrt{\left(\frac{4ac-b^2}{4c}\right)}$$

$$\times\cos(i.u\sqrt{c}) = a.\cos(i.u\sqrt{c}).$$

Now we have

$$\frac{dv}{du} = \frac{x^m}{(a+bx+cx^2)^n} = \left(-\frac{1}{2c}\right)^m \frac{[i.\beta \sin(i.u\sqrt{c})+b]^m}{[a.\cos(i.u\sqrt{c})]^n},$$

the integration of which can be performed by well known methods.

If, for example,

$$v = \int \frac{x^4 dx}{(a+bx+cx^2)^3 \sqrt{(a+bx+cx^2)}}$$

is required, we have

$$\frac{dv}{du} = \frac{x^4}{(a+bx+cx^2)^3} = \frac{1}{(2c)^4} \frac{[i.\beta \sin(i.u\sqrt{c})+b]^4}{[a.\cos(i.u\sqrt{c})]^3},$$

which is easily integrated by well known methods.

When

$$v = \int \frac{dx}{x^m(a+bx+cx^2)^n \sqrt{(a+bx+cx^2)}}$$

is required, we take the auxiliary

$$u = \int \frac{dx}{x\sqrt{(a+bx+cx^2)}} = \frac{i}{\sqrt{a}} \operatorname{arc}\sin\left(i\frac{2a+bx}{x\sqrt{(4ac-b^2)}}\right),$$

which gives

$$\frac{1}{x} = \frac{-i\sqrt{(4ac-b^2)}}{2a} \sin(i.u\sqrt{a}) - \frac{b}{2a};$$

$$\frac{\sqrt{(a+bx+cx^2)}}{x} = \sqrt{\left(\frac{4ac-b^2}{4a}\right)} \cos(i.u\sqrt{a}),$$

and leads to very simple results when $m > 2n$.

I will now present one example of the "binomial" class of integrals to show the treatment without rationalization, which example I take from *Sohnke's* Collection of Problems. It is required to integrate

$$v = \int \frac{32.dx}{x^5(1-x)^{\frac{7}{2}}}.$$

When rationalizing in the usual way this function gives rise to a rather complicated expression for the integral. We take as auxiliary

$$du = \frac{dx}{x\sqrt{(1-x)}};\ \therefore u = 2\sin x^{\frac{1}{2}},\ \therefore x = \sin^2\tfrac{1}{2}u;\ \sqrt{(1-x)} = \cos\tfrac{1}{2}u.$$

$$\therefore \frac{dv}{du} = \frac{32}{x^4(1-x)^3} = \frac{32}{\sin^8\frac{1}{2}u.\cos^6\frac{1}{2}u}.$$

To integrate this last function in the simplest way we take

$$dy = \frac{du}{\sin^2\frac{1}{2}u};\ y = -\cot\tfrac{1}{2}u;\ \therefore \sin^2\tfrac{1}{2}u = \frac{1}{1+y^2};\ \cos^2\tfrac{1}{2}u = \frac{y^2}{1+y^2}.$$

$$\frac{dv}{dy} = 64(1+y^2)^6 \frac{(1+y^2)^6}{y^{12}};$$

$$v = 64\int \frac{(1+y^2)^{12}}{y^{12}} dy.$$

[To be continued.]